shooting (called "artillery method"), differences and discrete variational methods. An exposition of the numerics of elliptic partial differential equations leads to iterative methods for linear systems (including conjugate gradients); norms and condition numbers of matrices appear at this late stage. An economic coding of the associated sparse matrices is suggested. In connection with heat conduction problems there is an interesting analysis of the feasibility of a systematic increase of the time step due to the elimination of high frequency error components. A short discussion of the one-dimensional wave equation completes the numerical analysis of differential equations. In just the same fashion, the section on eigenvalue problems guides the student to an understanding of the essential numerical aspects and methods without confusing him by unnecessary technicalities. Even the treatment of the shifts in the *LR*-method remains transparent and immediately convincing.

A most interesting appendix "An axiomatic basis for numerical computing and its application to the *QD*-algorithm" concludes the book. Rutishauser's approach is not to introduce intricate algebraic structures but simply to formulate as axioms those properties of common computer arithmetic which are necessary for a rigorous analysis of algorithm on a digital computer. The feasibility of his approach is shown by the proof of theorems on the performance of the *QD*-algorithm which has been introduced in the first part of the appendix. (The manuscript of this appendix was not completed when the author died.)

It is hoped that this book will see widespread use by the students for whom it has been intended and for whom it would furnish an exquisite introduction into the subject. In any case, the editors have set a worthy memorial to their friend H. Rutishauser.

H. J. S.

8[4.10.4, 5.10.3, 5.20.4].—J. T. ODEN & J. N. REDDY, *An Introduction to the Mathematical Theory of Finite Elements,* Wiley, New York, 1976, xii + 429 pp. Price $21.95.

This book is devoted to the mathematical foundations of the Finite Element Method (F E.M.) and its application to the approximation of elliptic boundary value problems and time dependent partial differential equations.

Let us first describe briefly the content of each chapter: Chapter 1 is an introduction in which we find a brief history of the F.E.M., an outline of the following chapters and, finally, some of the mathematical notation to be used in the remaining part of the book.

The following eight chapters could be divided into two parts. Part I (Chapters 2, 3, 4, 5) contains the mathematical background for, and Part II (Chapters 6, 7, 8, 9) the theory of, the F.E.M. Each chapter has its own bibliography.

In Chapter 2 the authors define distributions on a domain Ω of $\mathbf{R}$, their derivatives, the convergence of a sequence of distributions, etc.; all these definitions are illustrated by various examples. Also distributional differential equations and the concept of fundamental solutions are briefly considered.

Chapter 3 is related to the theory of Sobolev spaces. Once the definition of such spaces has been given, the authors, following Sobolev [1], prove several properties of the Sobolev spaces, in particular various embedding theorems, when Ω has the so-called cone property.

In Chapter 4 the authors use the approach of Lions and Magenes [2] to define the the Sobolev spaces $H^s(\mathbf{R}^N)$, first for $s \in \mathbf{R}_+$, and then for $s < 0$; this approach which is now classical is based on the use of the Fourier transform. Then the authors prove a trace theorem for the functions of $H^m(\mathbf{R}^N_+)$ ($\mathbf{R}^N_+ = \{x \in \mathbf{R}^N \mid x = (x_1, x_2, \ldots, x_N), x_N > 0\}$) and study various properties of the trace operators (continuity, surjectivity, etc.). To define $H^s(\Omega)$ for $\Omega \neq \mathbf{R}^N$ and s arbitrary, the authors, following Lions and Magenes, loc. cit., use the theory of interpolation between Hilbert spaces. Then using the local mapping method and the above interpolation theory they prove trace theorems for $H^s(\Omega)$ with s not an integer.

Chapter 5 may be viewed as an introduction to the theory of elliptic boundary value problems. After some preliminary definitions and results the authors discuss results of existence, compatibility, uniqueness, for the general elliptic boundary value problem:

$$\begin{aligned} Au &= f \quad \text{in } \Omega, \\ B_k u &= g_k \quad \text{on } \partial\Omega,\ k = 0, 1, \ldots, m - 1, \end{aligned}$$

where A is an elliptic differential operator of order m and where B_k is a differential boundary operator of order k.

The second part of the book is devoted to the theory of the F.E.M.

In Chapter 6 a definition of finite elements is given, and the authors discuss several concepts such as connectivity, local and global representation; these various topics are illustrated by various examples and figures. Then following Aubin [3], the authors define restriction, prolongation and projection operators and discuss with various examples the concepts of conjugate basis functions.

The next part of the chapter is related to classical concepts such as Lagrange families of finite elements, Serendipity elements, Hermite elements, conforming elements, isoparametric curved elements; all these finite elements are of polynomial type. Then there is a brief discussion of the Wachspress [4] rational finite elements. These concepts are illustrated with two- and three-dimensional examples and various figures. Then the authors, following several papers of Ciarlet and Raviart, prove, via the Bramble-Hilbert Lemma, interpolation error estimates in the Sobolev norms which can be applied to straight finite elements. The more complicated case of curved finite elements is briefly discussed at the end of the chapter.

In Chapter 7 the authors consider variational boundary value problems of elliptic type and give several examples of such problems. They then define the concept of coercive bilinear forms and they give several examples to illustrate it. In the following sections the authors introduce, following Babuška, the concept of weakly coercive bilinear forms and they discuss the treatment of nonhomogeneous boundary conditions viewed as constraints through the use of Lagrange multipliers. The authors conclude this chapter by proving a generalized Lax-Milgram theorem for variational problems related to weakly coercive bilinear forms. This theorem contains the classical Lax-Milgram theorem (for coercive bilinear forms) as a special case.

Chapter 8 is, from the finite element point of view, the most important part of the book since it describes the use of the finite elements studied in Chapter 6 to approximate the variational problems of Chapter 7. In the first sections there is a general study of Galerkin approximations, and a general formula for the approximation error is derived. Finite element approximations are then viewed as practical methods to implement the method of Galerkin. Finite element subspaces are built and various properties of the corresponding approximations are given, for example error estimates in weaker norms using the Aubin-Nitsche method. The so-called Inverse Property is also discussed. In Section 8.6 general error estimates in various norms are proved and the numerical results obtained from a simple one-dimensional Dirichlet problem are analyzed and compared with the theoretical predictions.

There is then a brief discussion of $L^\infty(\Omega)$ estimates of the approximation error.

The one-dimensional case is studied in detail but the results of Nitsche and Scott for higher dimensions are stated without proof (they were not yet available at the time the manuscript was completed), furthermore a large number of references is given which may be used by any reader interested by these rather difficult topics.

In the following sections the authors discuss the influence on the approximation error of the quadrature and data errors and also the effect of the approximation of the boundary by a simpler one (of polygonal type in many cases). In Section 8.9 the authors describe the H^{-1} finite element approximations, introduced by Rachford and Wheeler. In this method which is well suited for stiff one-dimensional boundary value problems, one uses different spaces for the approximate solution (piecewise linear, discontinuous, for instance) and for the test functions (piecewise cubic, C^1 if the approximate solution is as above).

To conclude Chapter 8 the authors describe the hybrid and mixed finite element methods which seem to be becoming more and more popular these days. They restrict their study to the approximation of second order elliptic problems, but many references related to fourth order problems are given. Several kinds of mixed and/or hybrid approximations are considered and a priori error estimates are obtained.

Chapter 9, which is the final one of the book, may be viewed as an introduction to the approximation of time dependent problems.

The authors study first the effect of the space discretization on time dependent problems of diffusion type (for example, the heat equation). Assuming that there is no time discretization they obtain a system of ordinary differential equations in a variational form (semidiscrete L^2 Galerkin approximations) for which an error estimate in the $L^2(\Omega)$ norm is given at any time (relation (9.19)). To study the effect of the full discretization (time and space) the authors use a semigroup approach and obtain L^2 estimates of the approximation error.

In Section 9.6 the authors describe the full discretization of the standard wave equation by the ordinary explicit scheme. Stability conditions and errors estimates are given.

To conclude this chapter, the authors discuss briefly the approximation of some first order hyperbolic equations using the Laplace transform to obtain error estimates. This concludes the descriptive part of this review.

From a more critical point of view we would like to make four major observations concerning this book:

(1) We think that students or engineers with a modest mathematical preparation may find this book difficult since its mathematical level is fairly high.

(2) The methodology used in this book in studying the Sobolev spaces, follows the Soviet school approach. In particular, the embedding theorems are not proved for $C^k(\overline{\Omega})$, but for less standard spaces which we think are less useful than the $C^k(\overline{\Omega})$ spaces in the study of the convergence properties of the F.E.M. in many linear and nonlinear boundary value problems. In this direction we think that the Nečas [5] approach is better suited to the study of the mathematical properties of the F.E.M.

(3) From a practical point of view and thinking of the possible users, we regret that Theorem 6.8 of page 279, which is in fact the main result of the book, has not been illustrated by several examples to link it with the Lagrange and Hermite finite elements described previously.

(4) We think that Chapter 9 is too theoretical, for its small number of pages. In our opinion it would have been preferable to describe more schemes (multistep, Runge-Kutta, Newmark, Wilson, etc.), give the basic properties of them and just indicate the order of the approximation in the more standard cases.

During our review we noticed some minor mistakes or ambiguities:

P. 67: In Theorem 3.3,(iii) the relation

$$\int_\Omega D^\alpha uv\,dx = (-1)^{|\alpha|}\int_\Omega uD^\alpha v\,dw$$

holds for all $v \in W_q^m(\Omega)$.

Pp. 107–109: In Example 4.5, the Sobolev spaces $H^s(\Omega)$, s nonintegral, have not been defined yet.

P. 110: The dual space of $H^m(\Omega)$ is not contained in $H^{-m}(\Omega)$.

P. 113:

$$D_n^j u = \frac{\partial^j u}{\partial n^j} \equiv (-1)^j \frac{\partial^j u}{\partial t^j} = (-1)^j D_t^j u.$$

On lines 7 and 8 the normal derivatives of u and v of order $j < m$ match for any extension v of u.

P. 142: (4.106) (resp. (1.110)) are not true $\forall u \in H^m(\Omega)$ (resp. $\forall u \in H^s(\Omega)$).

P. 179: Example 5.20: Since $\partial\Omega = \{0, 1\}$, the notation $H^{s-3/2}(\partial\Omega)$ may be confusing for a beginner.

P. 319: The conventional Lax-Milgram theorem also applies to nonsymmetric strongly coercive bilinear forms.

P. 347: Example 8.4; since a piecewise linear function is not in H^2 in general, we do not understand the penultimate line of p. 347.

P. 370: We think that $L_2(P) = (L^2(\Omega))^2$.

P. 391: In these kinds of problems it is very important to specify to which spaces the initial value u_0 and the right-hand side f belong.

P. 396: If A is defined by (9.2) it would be important to specify U in this case, since A is obviously not bounded from $H^m(\Omega)$ into $H^m(\Omega)$.

To conclude our review we would like to say that J. T. Oden and J. N. Reddy have performed a considerable task in writing a mathematical introduction to the F.E.M. They have tried to make it as self-contained as possible and usable by beginners and people with a modest mathematical education, and have written a book with the following qualities:

. The general plan is remarkably well conceived.

. It gives an introduction to the mathematical foundations of the F.E.M. (as the title indicates).

. It is quite adequate for self-study for a mathematically oriented reader.

. It may be used as a text for an advanced Numerical Analysis course.

. It contains an extensive up-to-date bibliography.

R. GLOWINSKI

Université de Paris 6
U. E. R. 48,
Laboratoire Analyse Numérique
Paris, France

1. S. L. SOBOLEV, *Applications of Functional Analysis in Mathematical Physics*, Transl. Math. Monographs, Vol. 7, Amer. Math. Soc., Providence, R. I., 1963.

2. J. L. LIONS & E. MAGENES, *Non Homogeneous Boundary Value Problems and Applications*, Vol. 1, Springer-Verlag, New York, 1972.

3. J. P. AUBIN, *Approximation of Elliptic Boundary-Value Problems*, Wiley-Interscience, New York, 1972.

4. E. L. WACHPRESS, *A Rational Finite Element Basis*, Academic Press, New York, 1975.

5. J. NEČAS, *Les Méthodes Directes en Théorie des Équations Elliptiques*, Masson, Paris 1967.